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A Chaotic Map with a Flat Segment Can Produce a Noise-Induced Order

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Matsumoto and Tsuda studied the effects of noise on chaos in a one-dimensional Belousov–Zhabotinsky (BZ) map and found noise-induced order, that is, an external noise destroys a chaotic behavior and produces some kind of order (periodicities). This phenomenon is very interesting in understanding the relation between chaos and natural phenomena. The present paper proposes a unimodal piecewise linear map which has a flat segment. It is shown numerically that the noise-induced order can be observed in this simple map in the same way as the BZ map. These numerical results clarify the mechanism of noise-induced order.

KEY WORDS: One-dimensional mapping; noise-destabilized chaos; flat segment.

1. INTRODUCTION

A one-dimensional discrete time dynamical system (a one-dimensional mapping) has been extensively used to describe the temporal change of a quantity in natural phenomena.[1] Such a dynamical system, even though very simple, can exhibit rich dynamical structures, including chaos. Over the past 15 years, immense efforts have been invested in the study of chaotic dynamical systems. One would like to understand the role of chaos in natural phenomena. It is important to study the effects of an external noise on a chaotic dynamical system in understanding the relation between chaos and natural phenomena since all real systems are subjected to various noise.

In recent years, many chaotic dynamical systems have been investigated numerically and/or analytically in the presence of noise.[2,3,5]

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In these studies, one can distinguish two types of noise effects. In the first, an external noise induces the transition from the "ordered" (periodic) state to the chaotic state. Crutchfield *et al.*[4] studied the effects of fluctuations on a dynamical system which undergoes cascade bifurcations and showed that the presence of noise leads to a bifurcation gap. It is also shown that fluctuations increase the degree of randomness of chaos. Mayer-Kress and Haken[6] studied the dynamics of the logistic map in the presence of noise and observed the noise-induced transition from the stable periodic orbits to chaos.

The second type of noise effect is to induce the opposite transition: The chaotic state can be driven into the "ordered" state by an external noise. Mayer-Kress and Haken[7,8] studied the smooth perturbation of the logistic map which has a positive Schwarzian derivative and observed that the Lyapunov exponent changes its value from positive to negative by an external noise. Matsumoto and Tsuda[9] studied the Belousov–Zhabotinsky (BZ) map and observed an interesting phenomenon called a noise-induced order. In fact, an external noise killed the chaotic behavior and produced some kind of order (periodicities). They further studied the BZ map and showed various interesting phenomena on the noise-induced order. Properties of the noise-induced order they found are summarized as follows:

i. A sharp peak appears in the power spectrum by adding noise.[9]

ii. The Lyapunov exponent changes its value from positive to negative by adding noise.[9]

iii. The entropy decreases as the noise level increases.[9,10]

iv. The mutual information decreases exponentially in the presence of noise, while it decreases in an oscillatory way in the absence of noise.[12,13]

Matsumoto and Tsuda[9–12] investigated the dynamics of the BZ map in detail, taking note of the steep part of the map, and concluded that the noise-induced order is mainly caused by the steepness of the BZ map.

In this paper, I propose a unimodal piecewise linear map (the TWFS map) which does not have a steep part but has a flat part. Using this map, it is shown numerically that the noise-induced order with properties i–iv is observed as in the case of BZ map. It is also shown that a variance of the noise necessary to extinguish the chaos becomes smaller as the slope of the flat part decreases. Thus I consider the mechanism of the noise-induced order. I show that a flat part plays an important role in producing the noise-induced order.

The organization of the paper is as follows. Section 2 illustrates briefly the dynamics of the BZ map. Section 3 specifies the TWFS map to be considered. Section 4 calculates numerically the probability density, the power spectrum, the Lyapunov exponent, the entropy, and the mutual information for the TWFS map. The paper concludes with some brief comments in Section 5.

2. THE BZ MAP

Consider a one-dimensional discrete time dynamical system (a 1D mapping) of form

$$x_{n+1} = f(x_n), \qquad n = 0, 1,... \tag{1}$$

where f is a nonlinear function on the unit interval [0, 1]. Call x_n the nth state of the system (1) and call x_0 the initial state particularly. A set of successive points $\{x_n\}(n = 0, 1, 2,...)$ is called the orbit of the map $f(x)$ or the orbit of the system (1). Though an orbit of the system depends on its initial state, the physically interesting behavior of a system is what is observed after initial transient behaviors die away. The set of states which an orbit eventually visits is called the system's attractor.

Let us investigate the effects of an external noise on the dynamics of the system. Consider the following system:

$$\hat{x}_{n+1} = f(\hat{x}_n) + \xi_n, \qquad n = 0, 1,... \tag{2}$$

where ξ_n are the pseudo-random numbers generated by a computer. The random numbers are equally distributed in the interval $[-\delta/2, \delta/2]$. ξ_n has a zero mean and a standard deviation $\delta/2\sqrt{3}$. ξ_n is interpreted as an external noise to the system (1) and δ controls the amplitude of noise or the noise level. Note that the noisy system (2) includes the noiseless system (1) as the special case ($\delta = 0$) of the system (2).

Matsumoto and Tsuda investigated exclusively the case where $f(x)$ is the BZ map (Fig. 1):

$$f(x) = \begin{cases} \left\{ a + \left(x - \frac{1}{8} \right)^{1/3} \right\} \exp(-x) + b, & 0 \leqslant x < 0.3 \\ c \left\{ 10x \exp\left(\frac{-10x}{3} \right) \right\}^{19} + b, & 0.3 \leqslant x \leqslant 1 \end{cases} \tag{3}$$

$$a = 0.50607357, \qquad c = 0.121205692$$

The map has both a flat part and a steep part (a neighborhood of $x = 1/8$). They investigated the effects of noise on the BZ map; they studied the system (2) and showed various interesting numerical phenomena called noise-induced order, characteristic features of which were summarized in i–iv of Section 1.

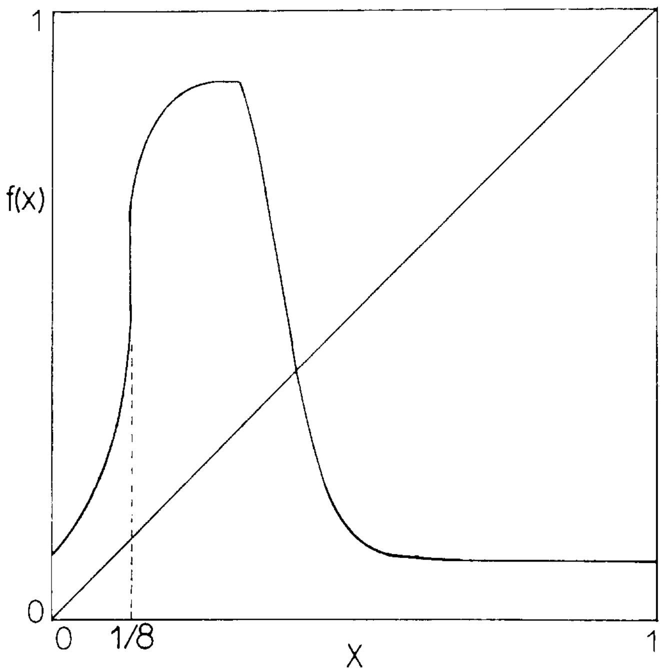


Fig. 1. Graph of the BZ map.

The remainder of this section investigates briefly the BZ map in order to get a heuristic description of the noise-induced order. The bifurcation diagram of Fig. 2 presents the change in the attractor of system (1) as a function of the bifurcation parameter b in the regime [0,0.025]. At a fixed bifurcation parameter, a periodic orbit consists of a set of separate points, while a chaotic attractor forms dense bands within the unit interval. In Fig. 2, the range of the bifurcation parameter b at which the orbit of the system is periodic is still wide and the parameter region at which the orbit is chaotic is narrow. Matsumoto and Tsuda chose a bifurcation parameter $b = 0.01213...$ where the behavior of the BZ map is chaotic and showed that the chaotic behavior is driven into the ordered state by adding noise.

As can be seen in comparing Eqs. (2) and (3), an addition of an external noise to the system is equivalent to causing the bifurcation parameter b to fluctuate around a mean value. At a fixed parameter b where the behavior of the system is chaotic, the system is influenced by adjacent periodic attractors by adding an external noise to the system. Roughly speaking, the effect of noise is to average the structure of deterministic attractors over some range of nearby parameters. It is expected that, if a

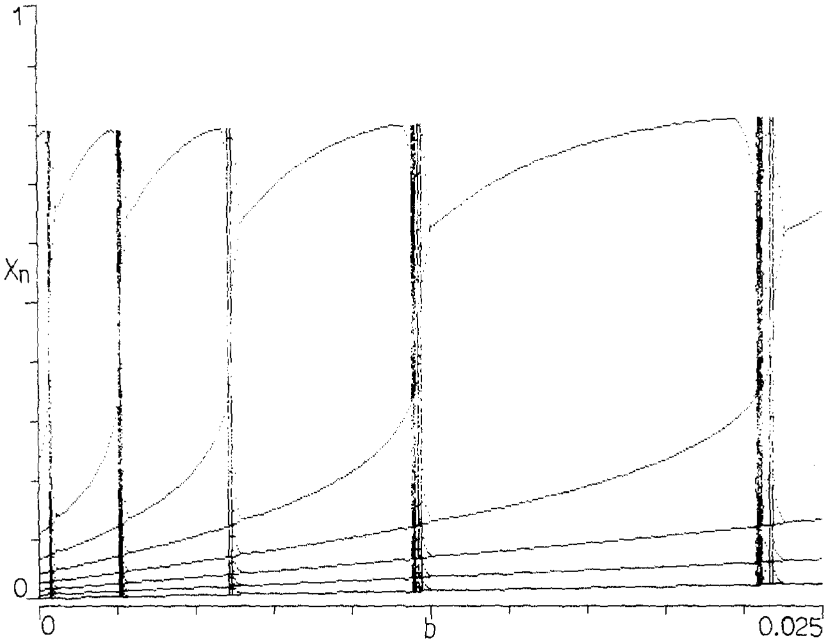


Fig. 2. Bifurcation diagram of the BZ map. Following a transient of 500 iterates, 2000 iterates of Eq. (1), namely $\{x_n\}$ ($i = 500, \ldots, 2500$) were recorded for each of 500 equally spaced b values on the interval $[0, 0.025]$.

chaotic parameter region is relatively narrower than nearby periodic parameter regions, the chaotic behavior is strongly influenced by nearby periodic orbits in the presence of noise and hence the noise-induced order appears. This suggests that various maps which have a similar bifurcation structure to the BZ map can produce the phenomenon of noise-induced order. I show this in the following sections.

3. THE TWFS MAP

Let us consider the following map:

$$f(x) = \begin{cases} 2x + b + \varepsilon, & 0 \leqslant x \leqslant \dfrac{a}{2} \\[2ex] 2(a - x) + b + \varepsilon, & \dfrac{a}{2} < x \leqslant a \\[2ex] \dfrac{\varepsilon(x - a)}{a - 1} + b + \varepsilon, & a < x \leqslant 1 \end{cases} \tag{4}$$

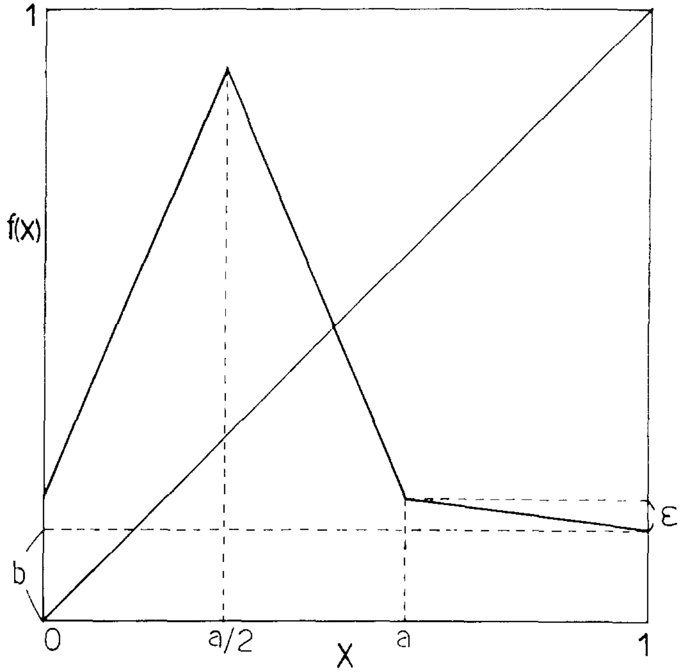
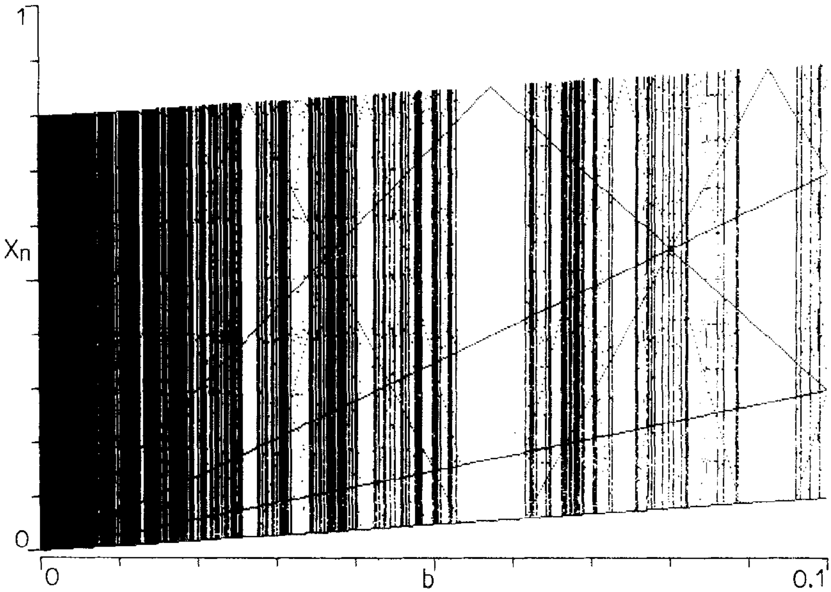


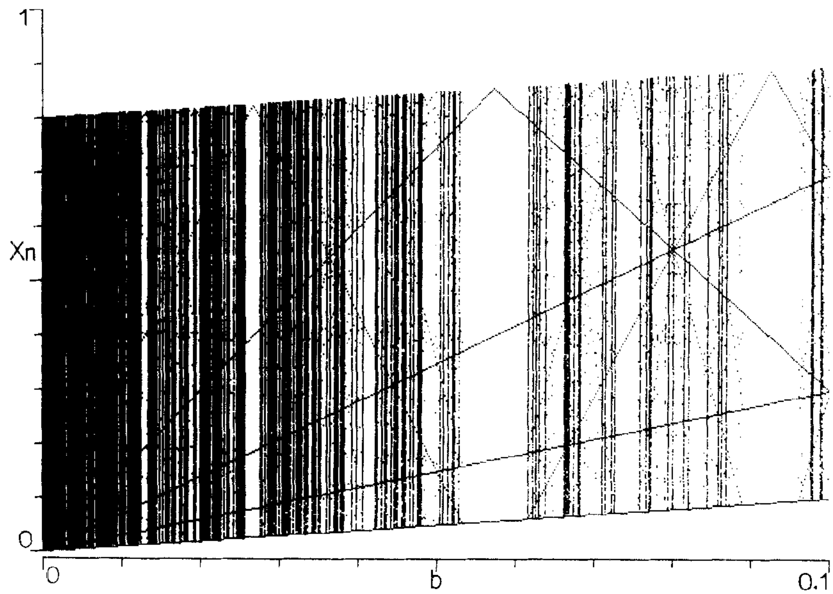
Fig. 3. Graph of the TWFS map.

Figure 3 illustrates the features of the map. This map is still simple because it consists of only a flat segment and a tent map, which is well known to be the simplest map exhibiting chaotic behavior. I call it the TWFS (tent with a flat segment) map for brevity. A parameter ε whose value is very small is used to express the flatness of the map. The TWFS map is considered to be the piecewise linear approximation of the BZ map. However, it should be noted explicitly that the TWFS map does not have a steep part, though the BZ map does.

Figure 4 shows the bifurcation diagram presents the change in the attractor of system (1) as a function of a bifurcation parameter b for fixed parameters a and ε. At a fixed bifurcation parameter value in the bifurcation diagram, a periodic orbit consists of a countable set of points, while a chaotic attractor fills out dense bands within the unit interval. In the interval $[0.06, 0.1]$ of the bifurcation parameter b, a chaotic parameter region is narrower than a periodic parameter region, while a chaotic parameter region is considerably wider in the interval $[0, 0.02]$. Comparing Fig. 4a with Fig. 4b, it is seen that the region of the parameter b at which the behavior of the system (1) is chaotic becomes narrower as ε decreases. Figures 4c and 4d are for the cases of $\varepsilon = 0$. Figure 4d is the magnification

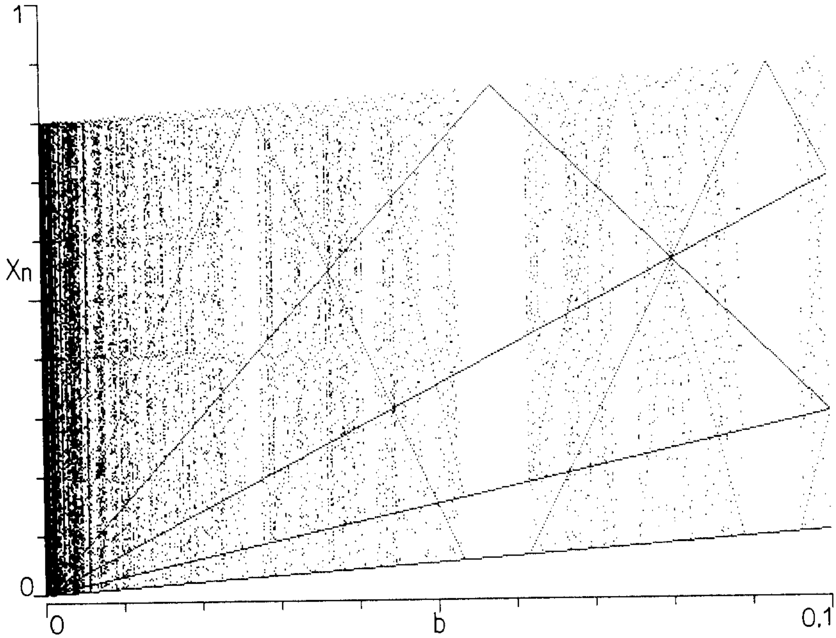


(a)

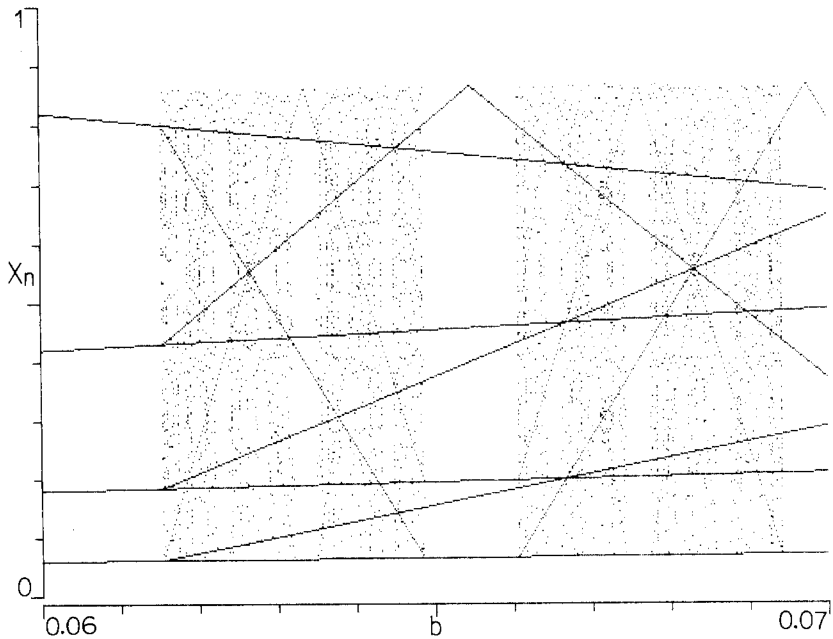


(b)

Fig. 4. Bifurcation diagrams of the TWFS map. Following a transient of 500 iterates, 2000 iterates of Eq. (1), namely $\{x_n\}$ $(i = 500, \ldots, 2500)$, were recorded for each of 500 equally spaced b values on the interval (a–c) [0, 0.1] and on the interval (d) [0.06, 0.07]. The value of a is 0.8 and the values of ε are: (a) 0.00001, (b) 0.000001, (c) 0, (d) 0. Part d is the magnification of the interval [0.06, 0.07] of part c. In the case of $\varepsilon = 0$, for almost all b values, orbits are periodic and a chaotic behavior does not appear, though the bifurcation has a very complicated nesting structure.



(c)



(d)

Fig. 4 (*continued*)

of the interval [0.06, 0.07] of the parameter b in Fig. 4c. In the case of $\varepsilon = 0$, the bifurcation diagram has a complicated nesting structure and various periodic orbits appear as the value of b changes. Note that a chaotic behavior does not appear at almost all b values in the case of $\varepsilon = 0$, though the bifurcation diagram has a very complicated structure.

As is seen from Figs. 4a–4d, the region of the bifurcation parameter b producing a chaotic behavior becomes narrower and the bifurcation structure approaches the case of $\varepsilon = 0$ as ε decreases. The global feature of the bifurcation diagram of the TWFS map is different from that of the BZ map. However, for a sufficiently small ε, the bifurcation structure of the TWFS map has a similar structure to the BZ map in the following sense: the TWFS map has a bifurcation structure in which the periodic parameter region is wide and the chaotic parameter region is very narrow. Therefore, it is expected that the noise-induced order can appear in the case of the TWFS map as in the case of the BZ map. We shall see this in the next section.

4. NOISE-INDUCED ORDER IN THE TWFS MAP

In this section I investigate the effects of noise on the system (1) numerically, using the TWFS map, and show that the noise-induced order with properties i–iv found by Matsumoto and Tsuda can appear. I set $a = 0.8$ throughout this section.

4.1. Probability Density

In the case of $\varepsilon = 0.000001$, I choose the bifurcation parameter $b = 0.08406$, so that the orbit of the system (1) become chaotic. In this case, the attractor of the system (1) forms dense bands within the unit interval rather than discrete points (see Fig. 4b). Therefore, one needs to consider the action of the system (1) on a probability distribution. Let us calculate the probability density (the invariant density) of the system (1) numerically.

Divide the unit interval into 500 equal segments. Let $P(i)$ denote the probability of finding the state x_n in the ith segment. The probability density $P(i)$ ($i = 1,..., 500$) is approximated by a histogram obtained from 10^5 iterates of Eq. (1). The probability density $P(i)$ ($i = 1,..., 500$) gives a frequency distribution of a trajectory starting from a certain initial state in the unit interval. If an external noise is added to the system (1), the probability density $P(i)$ is calculated using the iteration of Eq. (2) in the same manner.

The probability density $P(i)$ is shown in Fig. 5. Figure 5a shows the

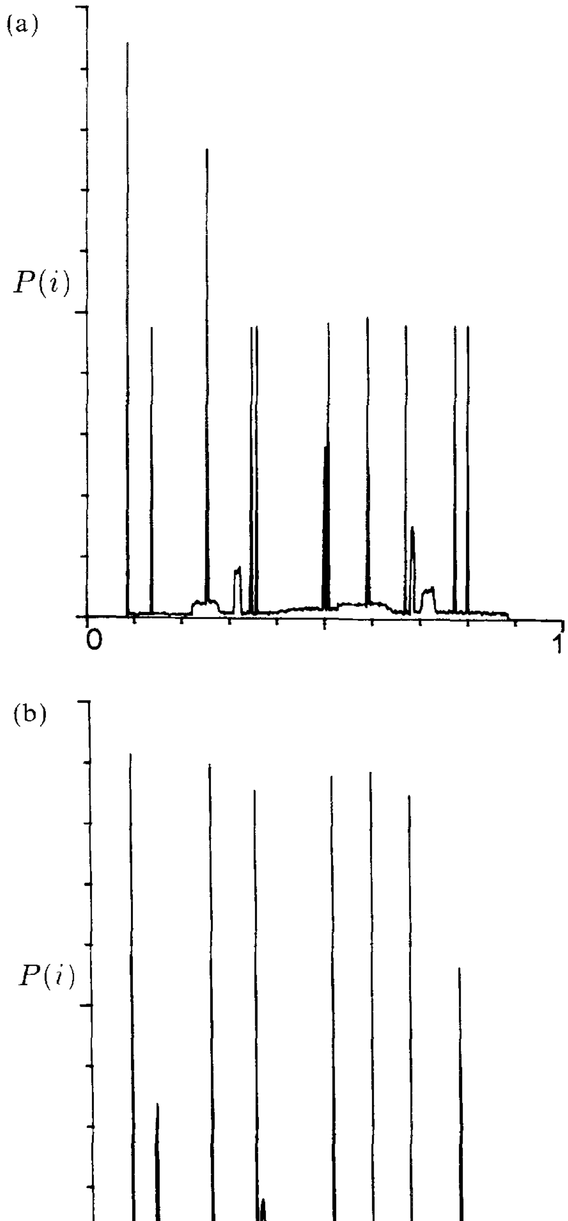


Fig. 5. Probability densities $P(i)$. The densities were computed by binning 10^5 iterates of Eq. (1) or (2) into 500 equally spaced bins following a transient of 500 iterates. The parameter values are $a = 0.8$, $b = 0.08406$. (a) The chaotic case ($\varepsilon = 0.000001$, $\delta = 0$). (b) The chaotic case with added noise ($\varepsilon = 0.000001$, $\delta = 0.00004$). (c) The periodic case with added noise ($\varepsilon = 0$, $\delta = 0.00004$).

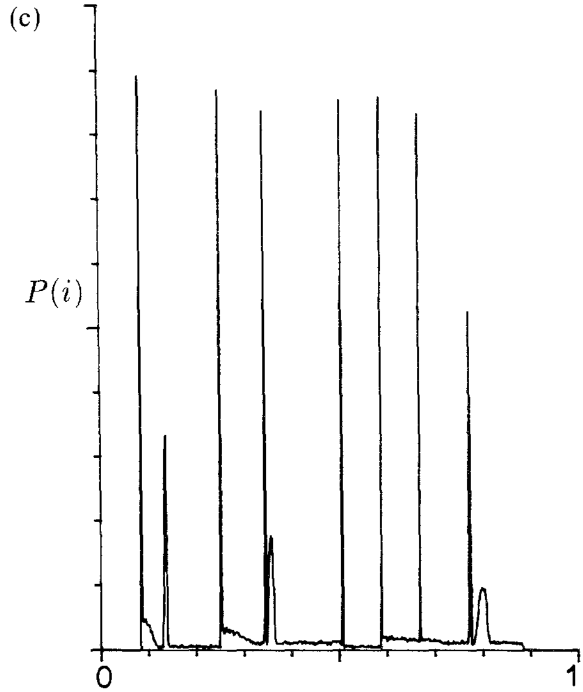
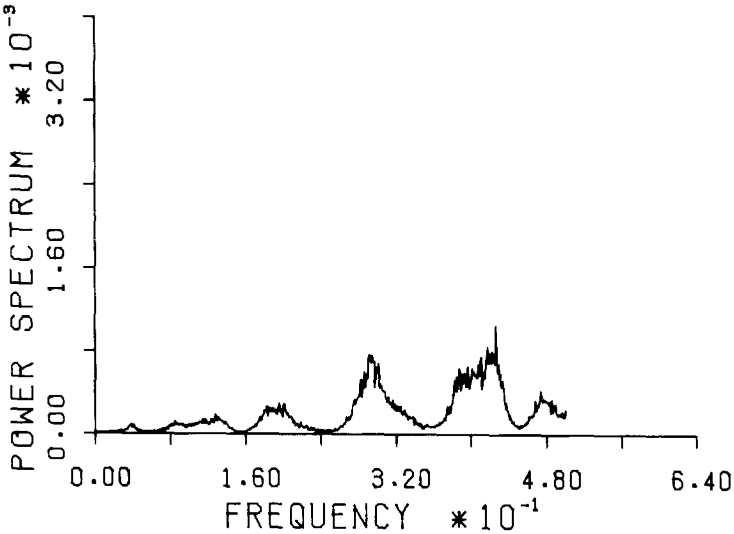


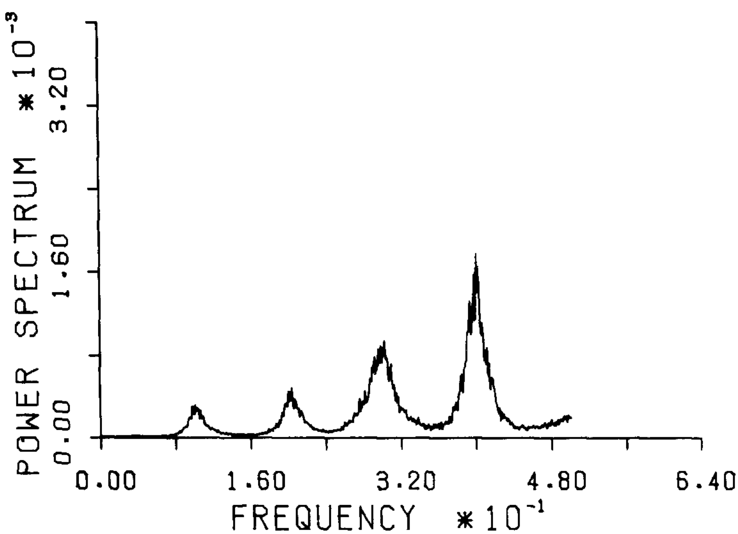
Fig. 5 (*continued*)

probability density in the noiseless case ($\delta = 0$), while Fig. 5b shows the noisy case ($\delta = 0.00004$). In Fig. 5a, there are several sharp spikes in the probability density and hence the chaos is considered to have some kind of periodicity. Comparing Fig. 5a with Fig. 5b, it is seen that the number of sharp spikes decreases and the amplitudes of the spikes become higher by adding noise. The chaotic orbit visits specific segments more frequently in the presence of noise than in the absence of noise. The chaotic orbit in the presence of noise has a stronger periodicity than the chaotic orbit without noise. This suggests that an external noise decreases the degree of randomness of the chaotic orbit in the dynamics. In this sense, some kind of order induced by an external noise appears.

Figure 5c is also the noisy case for $\varepsilon = 0$. As stated in Section 3, the orbit of the noiseless system (1) is not chaotic, but periodic in the case of $\varepsilon = 0$. In this case, the effect of noise is to average the periodic orbits at adjacent parameter values. Comparing Fig. 5b with Fig. 5c, it is seen that the both figures are very similar. This means that the attractor of the noisy chaotic system can be approximated by an appropriately weighted average of periodic orbits of the noiseless system (1) over some range of nearby

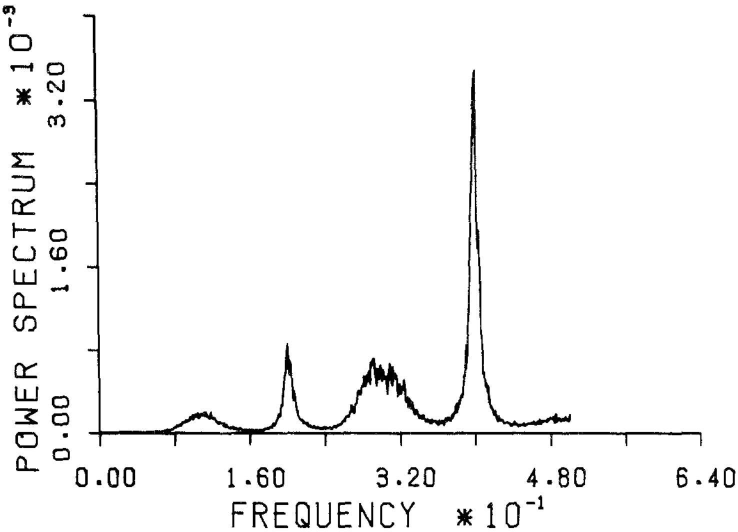


(a)



(b)

Fig. 6. Power spectral densities of the orbits $\{\hat{x}_n\}$ of the TWFS map for $a = 0.8$, $b = 0.08406$, $\varepsilon = 0.000001$, and various values of δ: (a) 0, (b) 0.00001, (c) 0.00004, (d) 0.01. The spectral densities were computed using the average of FFT of 50 sequences with 1024 points. For a critical noise level $\delta = 0.00004$ (panel c), the power spectral density has a strong periodicity.



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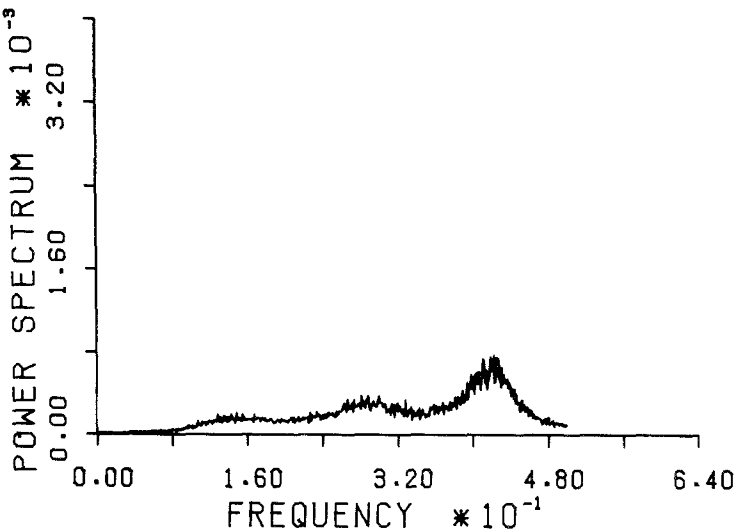


(d)

Fig. 6 (*continued*)

parameters. The noise-induced order is considered to have a close connection with the bifurcation structure of the periodic orbits of the system (1); if the chaotic parameter region is much narrower than the periodic parameter region, the chaos is strongly influenced by adjacent periodic orbits and hence the noise-induced order can appear.

4.2. Power Spectral Density

I calculated numerically the power spectral densities of the orbit $\{\hat{x}_n\}$ using the fast Fourier transform. The results are shown in Fig. 6. Figure 6a shows the power spectral density in the case without noise. The noise level δ is gradually increased from Fig. 6b to Fig. 6d. In Fig. 6a, there are several peaks in the power spectral densities. It is also shown that the chaotic orbit has some kind of periodicity, as shown in Fig. 5a. If the noise level δ is increased, the amplitudes of sharp peaks become higher. For the critical noise level $\delta = 0.00004$, the amplitudes of the sharp peaks take maximum values. If the noise level δ further increases, the sharp peaks disappear.

In Fig. 6, it is shown that some kind of order (periodicity) appears for a critical noise level. This result is consistent with the previous calculation of the probability density in Section 4.1.

4.3. Lyapunov Exponent

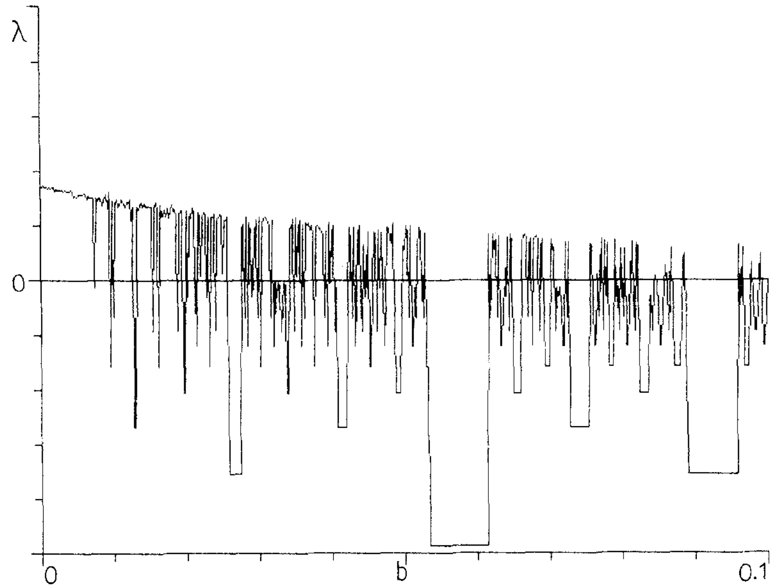
In this subsection, I calculate the other indicator for a chaotic behavior of the system and study how the chaotic characteristic of the system changes by adding noise.

The Lyapunov exponent λ is a quantity which measures the average divergence of orbits starting from nearby initial states. The Lyapunov exponent λ of the system (1) is defined by

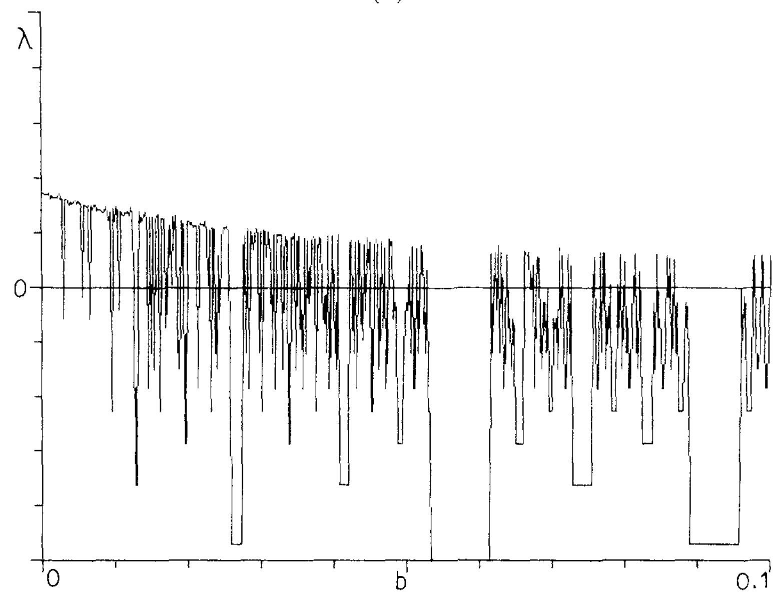
$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln \left| \frac{df}{dx}(x_k) \right| \tag{5}$$

When an external noise is added to the system (1), the Lyapunov exponent λ of the system (2) can be defined in close analogy to the formula (5) if we replace the sequence $\{x_n\}$ by $\{\hat{x}_n\}$. Here $\lambda < 0$ means that the orbit $\{x_n\}$ is stable and periodic, while $\lambda > 0$ means that the orbit $\{x_n\}$ is locally unstable and chaotic.

Figure 7 illustrates how the Lyapunov exponent λ changes its value as the bifurcation parameter b changes. Both Figs. 7a and 7b are noiseless cases ($\delta = 0$); for Fig. 7a, $\varepsilon = 0.00001$; for Fig. 7b, $\varepsilon = 0.000001$. From Figs. 7a and 7b, it is seen that the chaotic parameter region (the $\lambda > 0$

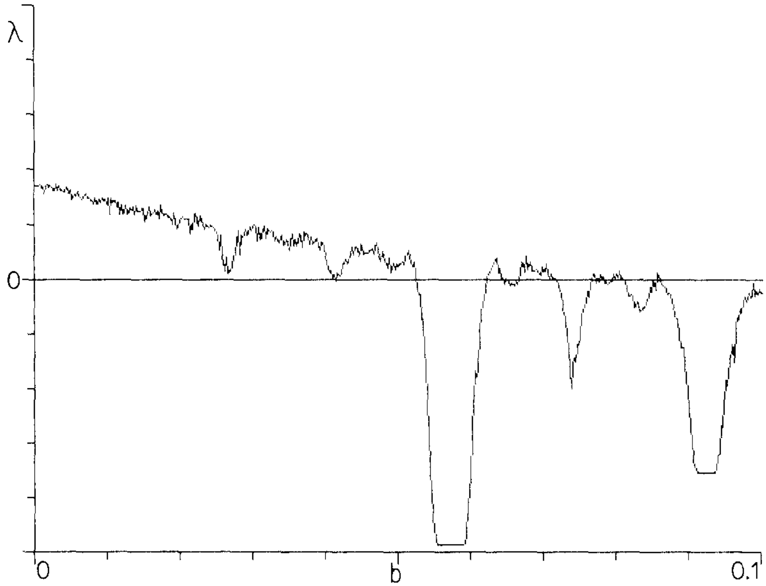


(a)



(b)

Fig. 7. Lyapunov exponent λ vs. the bifurcation parameter b. Here λ is calculated to within 1% using Eq. (5) at each b value. The parameter values are $a=0.8$ and (a) $\varepsilon=0.00001$, $\delta=0$, (b) $\varepsilon=0.000001$, $\delta=0$, (c) $\varepsilon=0.00001$, $\delta=0.005$, (d) $\varepsilon=0.000001$, $\delta=0.005$. Parts c and d are the noisy cases of the noiseless cases in parts a and b, respectively.



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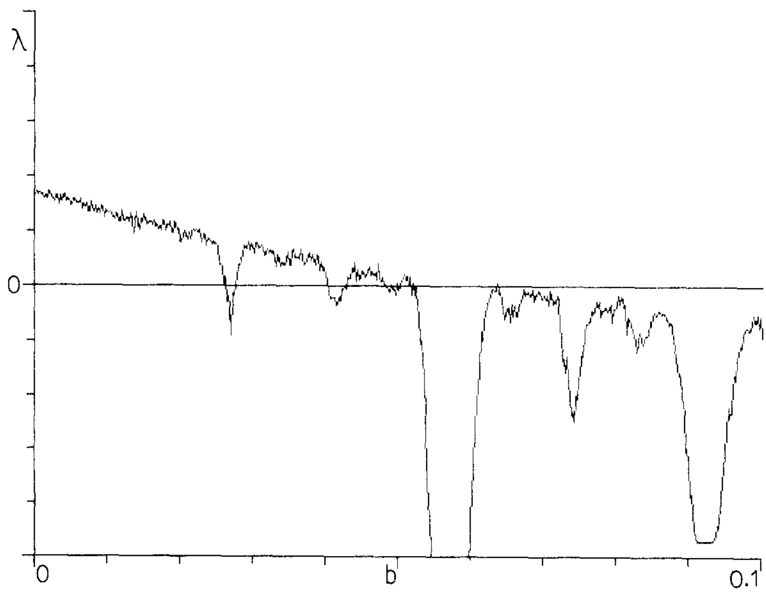


(d)

Fig. 7 (*continued*)

region) becomes narrower as ε decreases. This is also shown in Figs. 4a and 4b, where Figs. 7a and 7b correspond to Figs. 4a and 4b, respectively.

Figures 7c and 7d are the noisy cases ($\delta = 0.005$) corresponding to the noiseless cases of Figs. 7a and 7b, respectively. Comparing Fig. 7b with Fig. 7d, it is seen that λ changes its value from positive to negative for almost values of b in the interval [0.06, 0.1] by adding noise. In the sense of Lyapunov exponent, the noise-induced transition from chaos to order occurs. If the slope of the flat segment is large, the situation is different. Comparing Fig. 7a with Fig. 7c, one sees that the value of λ does not change from positive to negative though λ decreases its value in the interval [0.06, 0.1] of b. If the noise level δ is sufficiently high, λ also changes its value from positive to negative in this case. This means that the noise level δ necessary to extinguish the chaos becomes smaller as ε decreases.

Figure 8 illustrates how the Lyapunov exponent changes its value as the noise level δ increases for a fixed bifurcation parameter b. As the noise level increases, the Lyapunov exponent λ decreases. λ changes its value from positive to negative at a certain noise level. This means that the chaotic orbit is influenced more strongly by periodic orbits at nearby bifurcation parameters as the noise level δ increases.

For a large noise level δ, for example $\delta = 0.01$, the Lyapunov exponent takes a negative value, but the power spectral density does not have a

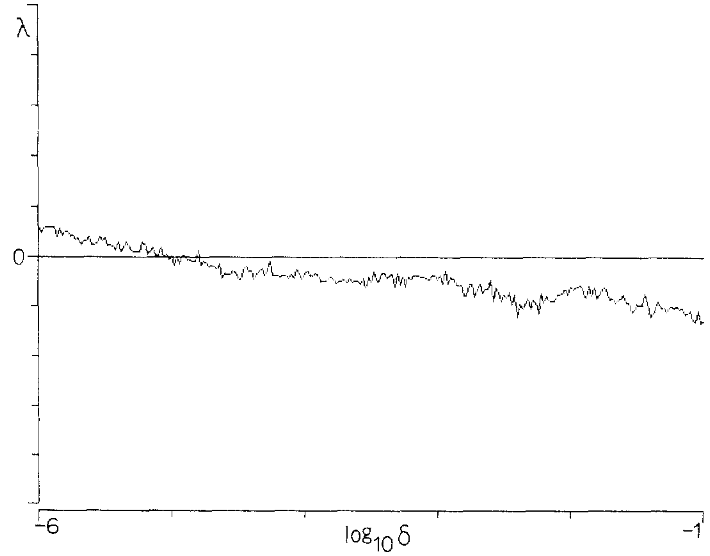


Fig. 8. Lyapunov exponent λ vs. the noise level δ. The parameter values are $a = 0.8$, $b = 0.08406$, $\varepsilon = 0.000001$. Here λ decreases as the noise level δ increases.

periodicity (see Fig. 6d). This means that the Lyapunov exponent is not necessarily an appropriate indicator for a chaotic behavior in the presence of large noise, as often pointed out.

In this subsection, using the Lyapunov exponent, we have seen that the chaos is destroyed by an external noise for sufficiently small ε and that the noise level δ necessary to extinguish the chaos becomes smaller as ε decreases. This suggests that the flatness of the TWFS map plays an important role in producing the noise-induced order.

4.4. Entropy

Section 4.1 has shown qualitatively that an external noise decreases the degree of randomness of chaotic orbits using probability distribution. In this subsection, we shall see this quantitatively using entropy.

Divide the unit interval [0, 1] into 500 equal segments. For this partition, define the entropy H of the system (1) or (2) as follows:

$$H = -\sum_{i} P(i) \log P(i) \tag{6}$$

where $P(i)$ is the probability distribution of an orbit at a given parameter value, such as shown in Fig. 5. Figure 9 illustrates how the value of H changes as the noise level δ increases. The entropy H takes a minimum value between $\delta = 1 \times 10^{-5}$ and $\delta = 1 \times 10^{-4}$. The result shows that some

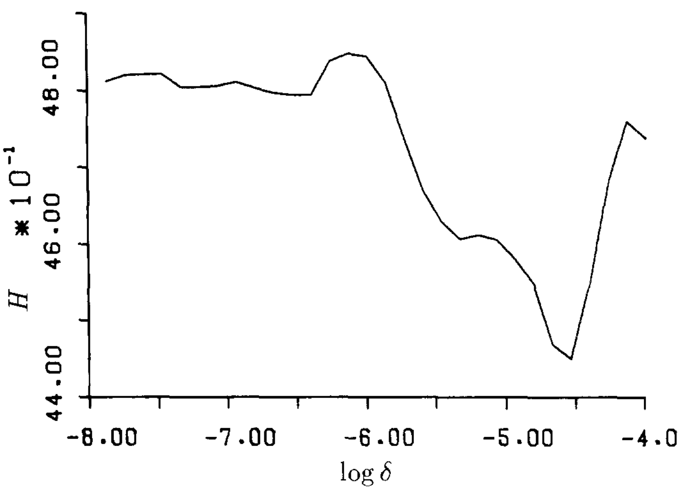


Fig. 9. Entropy H vs. the noise level δ. The values of the parameters are the same as Fig. 8. H takes a minimum value between the noise level $\delta = 1 \times 10^{-5}$ and $\delta = 1 \times 10^{-4}$.

kind of order appears at a critical noise level. This result is same as the result obtained by Matsumoto and Tsuda[9] using the BZ map.

Matsumoto[10] further studied the dynamics of the BZ map using a symbolic dynamical representation in order to clarify the mechanism of the noise-induced order. He regarded the system (2) as the nth-order Markov information source and calculated the entropy of the Markov source. It is shown that this entropy also decreases at a critical noise level. I shall calculate this entropy for the TWFS map and show that the same result can be obtained for the TWFS map.

Let us divide the unit interval $[0, 1]$ into two segments, $[0, a/2]$ and $(a/2, 1]$, labeled with symbols L and R, respectively. Thus, the time evolution $\{\hat{x}_0, \hat{x}_1, \hat{x}_2, \ldots\}$ of the system (2) is then translated into a sequence of symbols labeling the partition elements visited by an orbit,

$$s = \{s_0, s_1, s_2, \ldots\}, \qquad s_i = R, L \tag{7}$$

We consider the symbol sequence $s = \{s_0, s_1, s_2, \ldots\}$ as the product of an nth-order Markov source. Namely, we approximate the symbol sequence $s_0, s_1, s_2, \ldots$ by an nth-order Markov process. Let us define the entropy H_n of the nth-order Markov source:

$$H_n = -\sum p(s_1, \ldots, s_{n+1}) \log p(s_{n+1}/s_1, \ldots, s_n) \tag{8}$$

where the summation is taken over all possible sequences $s^{n+1} = (s_1, \ldots, s_{n+1})$, $s_i = R, L$. Here $p(s_1, \ldots, s_{n+1})$ is the probability of the occurrence of a sequence $(s_1, \ldots, s_{n+1})$ and $p(s_{n+1}/s_1, \ldots, s_n)$ is the conditional probability of finding a symbol s_{n+1} after observing a sequence $(s_1, \ldots, s_n)$ in the symbolic dynamics. Figure 10 illustrates how H_n changes as a function of the noise level δ. It is seen that the function H_n of δ converges as n increases. The symbolic dynamics of the system (2) is well approximated by the 10th-order Markov process and the entropy of Markov information source decreases at a suitable noise level.

Matsumoto[10] claimed that noise can suppress narrow states from symbolic dynamics if the widths of symbolic dynamical state (Markov partition) are nonuniform and hence the decrease of the entropy H_n occurs. The nonuniformity of the widths of symbolic dynamical states results from the steepness of a map. However, the TWFS map does not have a steep part, though it has a flat part. This, opposed to Matsumoto,[10] also suggests that a flat part plays an important role in the phenomenon of decrease of the entropy by adding noise.

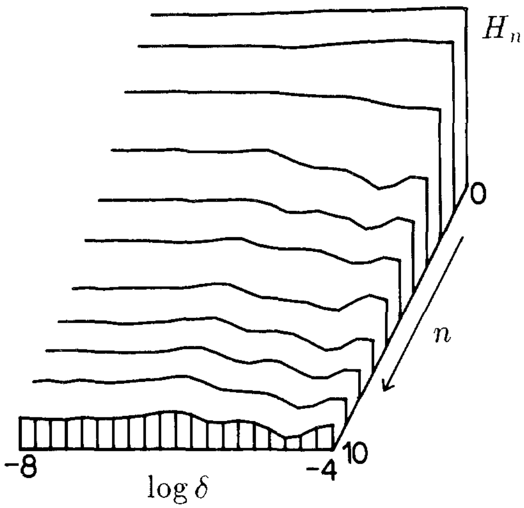


Fig. 10. Entropy H_n vs. the noise level δ for $n = 0, 1, \ldots, 10$. Here H_n is calculated numerically using the formula (8), where $p(s_1, \ldots, s_{n+1})$ and $p(s_{n+1}/s_1, \ldots, s_n)$ are approximated using 10^5 iterates of Eq. (2). The values of the parameters are the same as in Fig. 8. The entropy H_n also takes a minimum value at a certain noise level δ as in Fig. 9.

4.5. Mutual Information

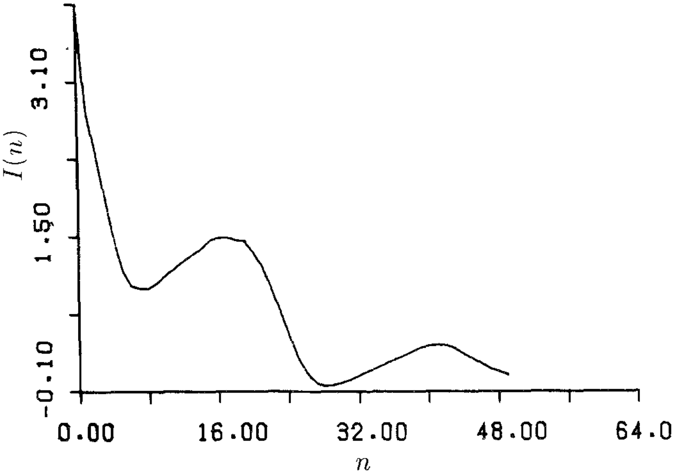
In order to clarify the real mechanism of the noise induced-order, Matsumoto and Tsuda[12,13] discussed the informational structure of a one-dimensional map. Namely, they calculated the mutual information of the BZ map and showed the phenomenon with the property iv stated in Section 1. Thus, they discussed the mechanism of the noise-induced order and insisted that the existence of the phenomenon of noise-induced order can be checked from the feature of the graph of the mutual information.

In this subsection, I calculate the mutual information for the TWFS map and present the same phenomenon as property iv in the BZ map.

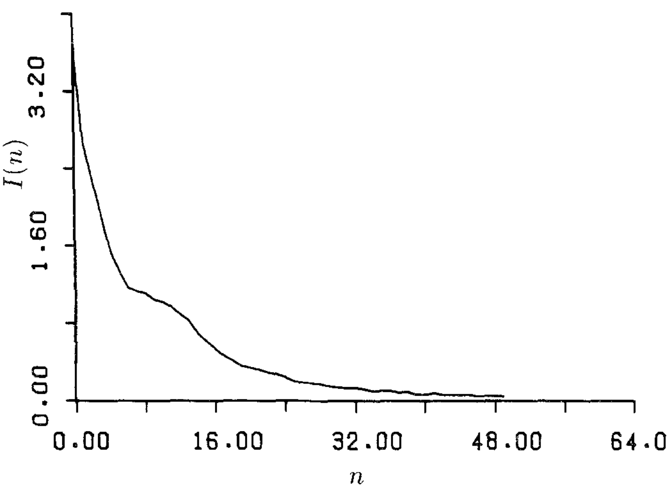
Divide the unit interval [0, 1] into 500 equal segments. For this partition, define the mutual information $I(n)$ as follows:

$$I(n) = -\sum_{i} P(i) \log P(i) + \sum_{i,j} P(i)\, P_n(j/i) \log P_n(j/i) \tag{9}$$

where $P(i)$ is defined such as shown in (6) and $P_n(j/i)$ is the conditional probability of a point starting the ith segment falling in the jth segment after n iterations. Figure 11 shows the mutual information $I(n)$. It is seen that the mutual information $I(n)$ of the noiseless system (1) decreases in an



(a)



(b)

Fig. 11. Mutual information $I(n)$ $(n = 0, \ldots, 50)$ for $a = 0.8$, $b = 0.08406$, $\varepsilon = 0.000001$, and for different values of δ: (a) 0, (b) 0.001. (a) The noiseless case. The mutual information $I(n)$ decreases in an oscillatory way. (b) The noisy case. $I(n)$ decreases exponentially. Here $I(n)$ is calculated using the formula (9). In this calculation, one starts with 1000 points uniformly distributed over each segment of the partition and approximates $P_n(j/i)$.

oscillatory way and that $I(n)$ decreases exponentially in the presence of noise. This result is in full agreement with the result[12] obtained for the BZ map.

5. DISCUSSION

This paper has presented the very simple example of a one-dimensional map (the TWFS map) which produces the phenomena of noise-induced order found by Matsumoto and Tsuda[9-12] using the BZ map. I calculated the probability density, the power spectrum, the Lyapunov exponent, the entropy, and the mutual information numerically for the TWFS map. In these numerical calculations, all the properties i–iv of the noise-induced order have been observed for the TWFS map. It is also shown that the noise level necessary to extinguish the chaos becomes smaller as the slope of the flat segment decreases.

The BZ map has both a wide, flat part and a steep part (see Fig. 1). Matsumoto and Tsuda insist that the noise-induced order is mainly caused by the steepness of the map. The TWFS map does not necessarily have a steep part, though it has a flat part. This, opposed to Matsumoto and Tsuda, suggests that the flat part in the map may play an important role in producing the noise-induced order.

Since the purpose of this paper is to present a simple example producing the noise-induced order, I calculated various statistical quantities only numerically and have not investigated the statistical quantities in detail. This detailed investigation is now in progress. However, the numerical calculations sufficiently emphasize a role of a flat part in the noise-induced order.

Section 3 showed that the TWFS map has a very complicated bifurcation structure only numerically. It is possible to determine the bifurcation structure of periodic orbits analytically, though it is difficult to investigate the statistical property of chaotic orbits at every bifurcation parameter b. I will show this analysis in the another paper.[14]

As seen in Section 4.1, the probability density of the chaotic case (the case of $\varepsilon \neq 0$, Fig. 5b) with added noise is very similar to that of the periodic case (the case of $\varepsilon = 0$, Fig. 5c) with added noise. I introduce the following intuitive discussion in order to understand why this phenomenon occurs. As described in Section 2, an addition of an external noise is equivalent to causing the bifurcation parameter b to fluctuate around a mean value. For sufficiently small ε, a chaotic parameter region is fairly narrower than nearby periodic parameter regions. Thus, the chaotic behavior is smeared and is influenced considerably by nearby periodic orbits in the presence of noise. I shall make this a little more precise. Let $P_{\varepsilon,b}(x)$ and $\tilde{P}_{\varepsilon,b}(x)$

[I denoted both by $P(i)$ in Section 4.1] be the probability density of the system in the absence of noise and that in the presence of noise, respectively. Approximate $\tilde{P}_{\varepsilon,b}(x)$ by the average of the probability densities corresponding to periodic orbits:

$$\tilde{P}_{\varepsilon,b}(x) \approx \frac{1}{n} \sum_{i=1}^{n} P_{0,b+\xi_i}(x) \tag{10}$$

where ξ_i is a random noise such as shown in (2). Since $P_{0,b+\xi_i}(x)$ is the probability density of the system without noise in the case of $\varepsilon = 0$, the system is periodic and $P_{0,b+\xi_i}(x)$ consists of several spikes. If the formula (10) gives a good approximation for a sufficiently large n, then the chaotic system in the presence of noise is considered to be affected considerably by the periodic behavior of the system at nearby bifurcation parameters. $P_{0,b+\xi_i}(x)$ denotes the stationary behavior of the system for a fixed bifurcation parameter $b+\xi_i$. Hence, it is necessary to take the transient behavior of the system into consideration for a more precise approximation of $\tilde{P}_{\varepsilon,b}(x)$. I am investigating along these lines.

In this paper, I investigated exclusively the TWFS map and showed the phenomenon of noise-induced order. However, noise-induced order can be observed in any one-dimensional map $f(x)$ which has a flat segment, for example $f(x) = ax \exp(-bx^2)$. A flat segment represents the stability of a system [a map $f(x)$]. A chaotic map with a flat segment is considered to be very stable and periodic as a whole, though the map may produce a chaotic behavior at a suitable bifurcation parameter. It is expected that one-dimensional maps with a flat segment appear in various studies of natural phenomena.

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